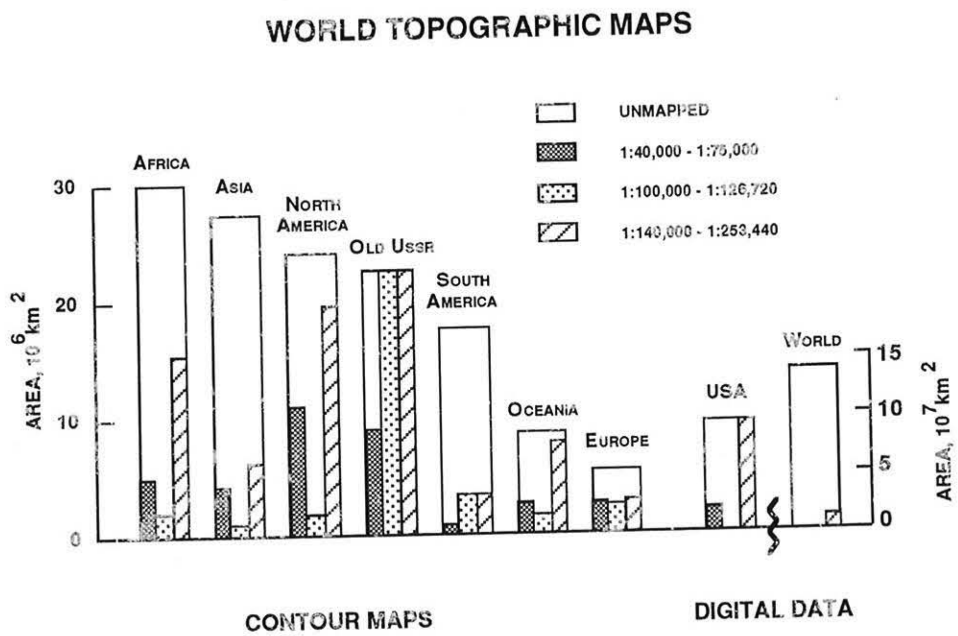
Measurement of Topography

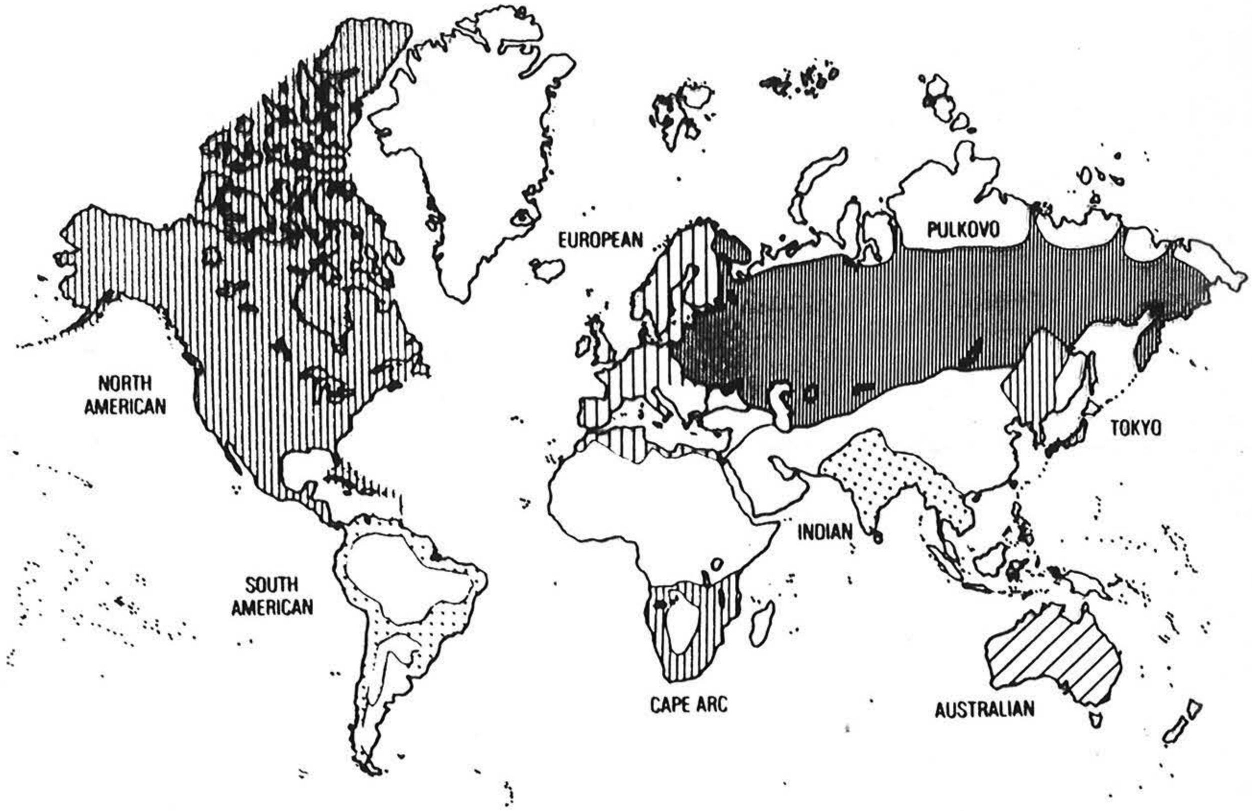
Now that we have demonstrated our ability to measure topography, let's look more in detail at the actual measurement of it. First, some rationale – why do we want to measure topography?

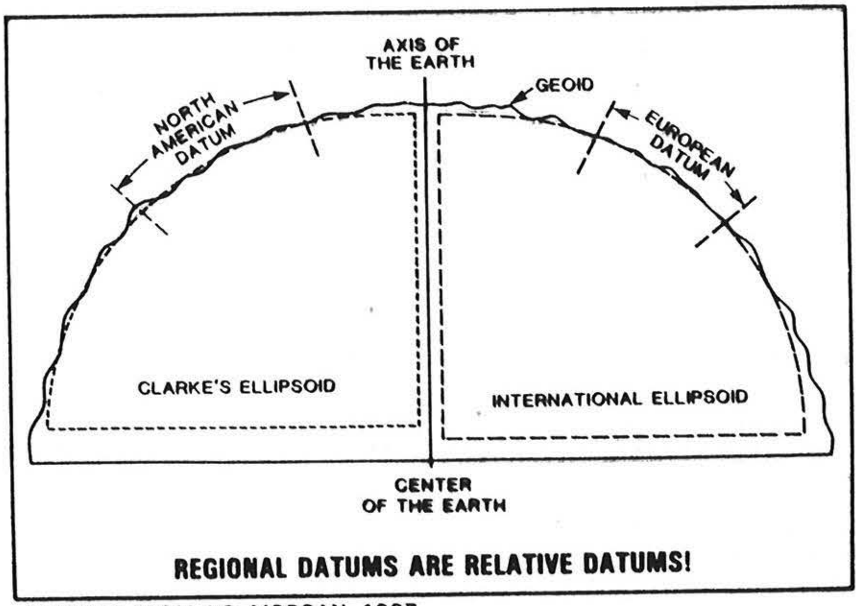
1. Not much of the world is well mapped, especially in digital form:



2. Different parts of the world are mapped on different "datums", or coordinate systems, making it difficult to implement global studies where values from one section of the Earth must be compared to those from other sections.

MAJOR GEODETIC DATUM BLOCKS





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Finally, let's take a look at what kind of accuracies we'd need for a variety of scientific studies:

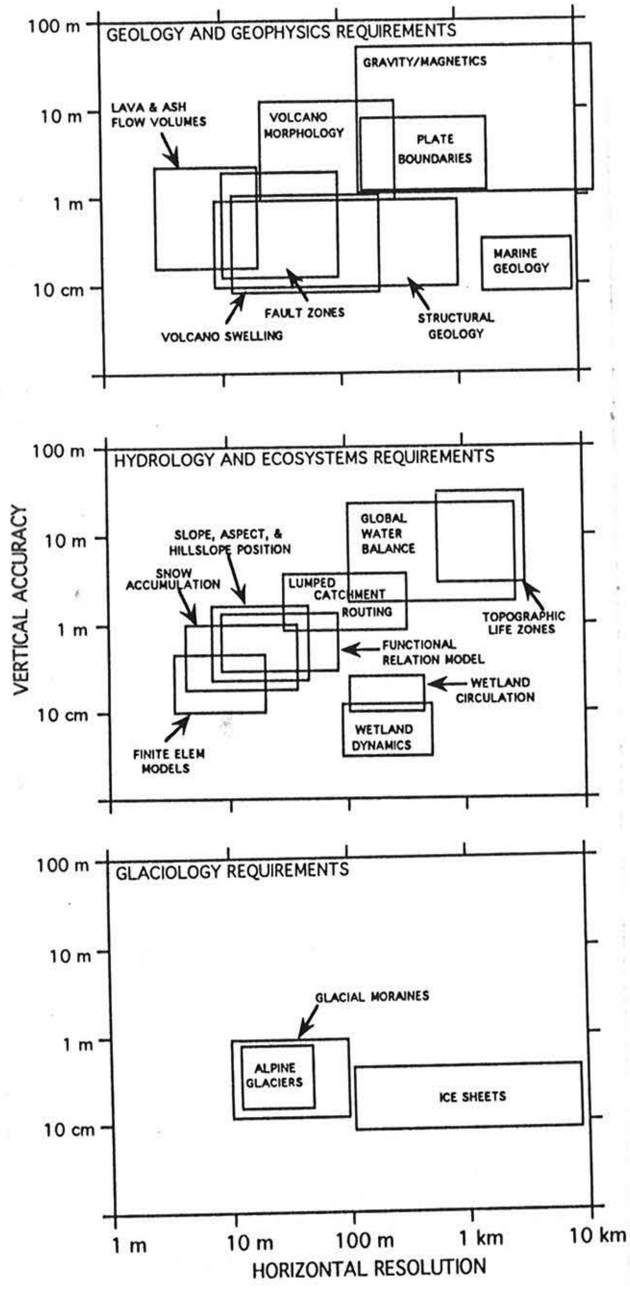
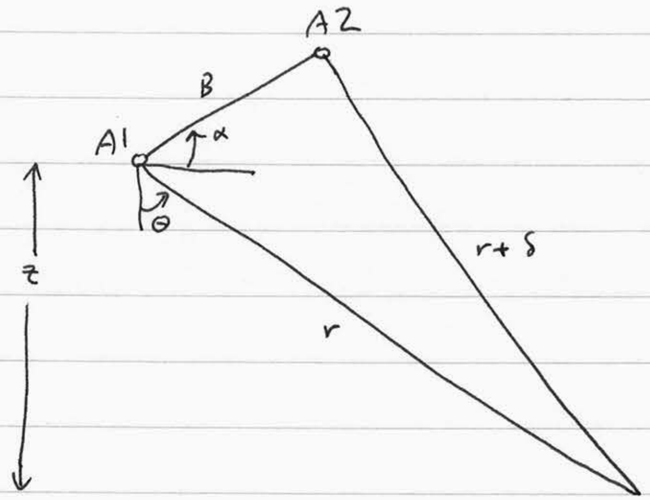


Fig. 1. Graphical depiction of horizontal and vertical topographic data accuracies required for several discipline studies. Each box represents a range of requirements for differing aspects of the studies.

Solution for Topography from Interferometric Phase

For our actual implementation of the topographic interferometer, we'll will rederive the phase vs. surface height equation without making the parallel-ray approximation, as the exact solution will be called for in precision applications. Again consider our construction:



We'll use previous definitions for all of the above quantities. From the law of cosines,

$$(r+\delta)^2 = r^2 + B^2 - 2rB\sin(\theta-\alpha)$$

from which

$$\sin(\theta-\alpha) = \frac{r^2 + B^2 - (r+\delta)^2}{2rB}$$

Also,

$$\delta = \frac{-\lambda}{4\pi}\phi$$

and

$$z = r\cos\theta$$

Hence given B, α, and r we can solve for θ, which gives us z. As before, we can estimate our phase error from the derivatives. We want

$$\frac{\partial z}{\partial \phi} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial \delta}\frac{\partial \delta}{\partial \phi}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta} r\cos\theta = -r\sin\theta$$

For $\frac{\partial \theta}{\partial \delta}$:

$$\sin(\theta-\alpha) = \frac{r^2+B^2-(r+\delta)^2}{2rB}$$

$$\cos(\theta-\alpha)\,\partial\theta = -\frac{2(r+\delta)}{2rB}\,\partial\delta$$

$$\approx -\frac{\partial\delta}{B}$$

$$\therefore \frac{\partial\theta}{\partial\delta} = \frac{-1}{B\cos(\theta-\alpha)}$$

and

$$\frac{\partial\delta}{\partial\phi} = \frac{-\lambda}{4\pi}$$

so $$\frac{\partial z}{\partial \phi} = (-r\sin\theta)\left(\frac{-1}{B\cos(\theta-\alpha)}\right)\left(\frac{-\lambda}{4\pi}\right)$$

$$= -\frac{\lambda}{4\pi}\frac{r\sin\theta}{B\cos(\theta-\alpha)}$$

which is the same result as we had in the approximate case. In other words, the error estimate for the parallel-ray approximation was good.

Now, there is a second error source in addition to the phase error. The baseline may not be known well either in terms of length B, or more likely in terms of position α. These contributions to height error are

$$\frac{\partial z}{\partial B} = \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial B}$$

$\frac{\partial \theta}{\partial B}$:

$$\cos(\theta-\alpha)\,\delta\theta = \frac{\partial}{\partial B}\,\frac{r^2+B^2-(r+\delta)^2}{2rB}\,\partial B$$

$$= \frac{2r\left(r^2+B^2-(r+\delta)^2\right) - 2rB\cdot 2B}{4r^2B^2}$$

$$= \frac{2r\left(B^2-2r\delta-\delta^2\right) - 4rB^2}{4r^2B^2}\,\partial B$$

$$\approx -\frac{\delta}{B^2}\,\partial B$$

$$\frac{\partial \theta}{\partial B} = \frac{-\delta}{B^2\cos(\theta-\alpha)}$$

$$\frac{\partial z}{\partial B} = \frac{r\sin\theta\,\sin(\theta-\alpha)}{B\cos(\theta-\alpha)}$$

This term usually is quite small in practical systems. But another baseline error term, from uncertainty in orientation, can be quite large.

$$\frac{\partial z}{\partial \alpha} = \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial \delta}\cdot\frac{\partial \delta}{\partial \alpha}$$

We already found $\frac{\partial z}{\partial \theta} = -r\sin\theta$, and $\frac{\partial \theta}{\partial \delta} = \frac{-1}{B\cos(\theta-\alpha)}$.

So, we need $\frac{\partial \delta}{\partial \alpha}$

$$\sin(\theta-\alpha) = \frac{r^2+B^2-(r+\delta)^2}{2rB}$$

$$-\cos(\theta-\alpha)\,\partial\alpha = \frac{-2(r+\delta)}{2rB}\,\partial\delta$$

$$\approx -\frac{\partial\delta}{B}$$

$$\frac{\partial\delta}{\partial\alpha} = B\cos(\theta-\alpha)$$

Thus

$$\frac{\partial z}{\partial \alpha} = (-r\sin\theta)\left(\frac{-1}{B\cos(\theta-\alpha)}\right)(B\cos(\theta-\alpha))$$

$$= r\sin\theta$$

This term may be understood as an unknown "tilt" across the image. In other words we cannot distinguish a tilt in the interferometer itself from a slope on the surface.

Let's plug a few numbers into these to understand the contributions to error from each. We'll use the NASA TOPSAR airborne system as an example. Its parameters are

$r \approx 10$ km

$\theta \approx 30°$

$B = 1.5$ m

$\alpha = 63°$

$\lambda = 6$ cm

Using our formulas from above,

$$\text{Phase error} \Rightarrow \sigma_z = \frac{\lambda}{4\pi} \frac{r \sin\theta}{B\cos(\theta-\alpha)} \sigma_\phi$$

$$\text{Baseline length} \Rightarrow \sigma_z = \frac{r\sin\theta \tan(\theta-\alpha)}{B} \sigma_B$$

$$\text{Baseline angle} \Rightarrow \sigma_z = r\sin\theta\, \sigma_\alpha$$

So, we need to evaluate (or measure) the system uncertainties on the right hand sides above. Beginning with σ_ϕ, suppose the system has an SNR of 20 dB.

$$\sigma_{\phi,1\,look} = \frac{1}{\sqrt{SNR}} = 0.1 \text{ radians}$$

Now, this radar calculates about 10 looks in the interferogram before we estimate the height. We'll just use the following formula which is found in several papers (and by reference in today's other handout)

$$\sigma_{\phi,\,multilook} \approx \frac{\sigma_{\phi,\,1\,look}}{\sqrt{2 \cdot N}}$$

where N is the number of looks. Thus

$$\sigma_\phi \approx 1\tfrac{1}{4}^\circ \text{ or } 0.022 \text{ radians}$$

thus $\sigma_z = \frac{\lambda}{4\pi} \frac{r\sin\theta}{B\cos(\theta-\alpha)} \sigma_\phi = 0.\ 42m$

For baseline length, aircraft deformation limits σ_B to about 10^{-4} m,

$$\sigma_z = \frac{r\sin\theta\tan(\theta-\alpha)}{B}\sigma_B = 0.216 \text{ m}$$ ← half of phase error

And for orientation, $\sigma_\alpha \approx 0.01°$, so

$$\sigma_z = r\sin\theta\,\sigma_\alpha = 0.88 \text{ m}$$ ← twice the phase error

So, we see that uncertainty in orientation is the major source of error in a typical system.

Solution for height

We obtained a set of equations above to allow us to determine heights from the phase measurements. This means if we know our imaging geometry and the absolute phase at each point we can solve for the topography. This requires, though, that we have "unwrapped" the phase at each point in our image.

We are going to defer discussion of phase unwrapping for the time being and discuss an approximate method of solution for now.

Begin with our equation relating surface topography to δ_i through the look angle θ:

$$\sin(\theta-\alpha) = \frac{r^2+B^2-(r+\delta)^2}{2rB}$$

$$= \frac{r^2+B^2-r^2-2r\delta-\delta^2}{2rB}$$

$$\approx \frac{B}{2r} - \frac{\delta}{B} \quad \text{(neglecting term of } \delta^2\text{)}$$

Note that if we neglect also the term $\frac{B}{2r}$ we get the plane parallel relation:

$$B \sin(\theta - \alpha) = -\delta$$ ← minus sign in our current coordinates

Then since $z = r\cos\theta$,

$$\delta = -B \sin\left(\cos^{-1}\frac{z}{r} - \alpha\right)$$

$$= -B\left[\sin\left(\cos^{-1}\frac{z}{r}\right)\cos\alpha - \cos\left(\cos^{-1}\frac{z}{r}\right)\sin\alpha\right]$$

$$= -B\left[\sqrt{1-\frac{z^2}{r^2}}\,\cos\alpha - \frac{z}{r}\sin\alpha\right]$$

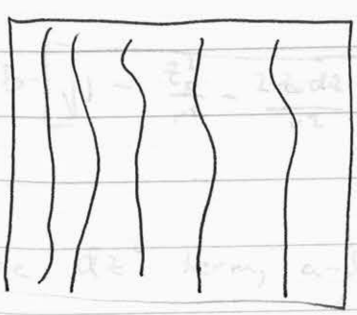
The phase we measure is then

$$\phi = \frac{4\pi}{\lambda} B\left[\sqrt{1-\frac{z^2}{r^2}}\,\cos\alpha - \frac{z}{r}\sin\alpha\right]$$

~~Expand the~~

This total phase contains the basic interferometer fringe pattern modulated by topographic variation. The "flat earth" pattern is simply that corresponding to $z = z_0$ everywhere.

This complicated-looking expression is simpler when viewed as an image:



← fringe pattern decreasing in spatial frequency with range, and modulated by topographic features.

Now, let's simplify and evaluate ϕ

$$\phi = \frac{4\pi B}{\lambda r}\left[\sqrt{r^2 - z^2}\cos\alpha - z\sin\alpha\right]$$

Expand about z_0 : $z = \underset{\substack{\uparrow \\ \text{average height}}}{z_0} + \underset{\substack{\uparrow \\ \text{topography}}}{dz}$

$$\phi = \frac{4\pi B}{\lambda r}\left[\sqrt{r^2 - z_0^2 - 2z_0 dz + dz^2}\cos\alpha - (z_0 + dz)\sin\alpha\right]$$

Neglecting dz^2 and making our usual approximation, (for $\sqrt{\ }$)

$$\phi = \frac{4\pi B}{\lambda r}\left[\sqrt{r^2 - z_0^2}\sqrt{1 - \frac{2z_0 dz}{r_0^2 + z_0^2}}\cos\alpha - z\sin\alpha - dz\sin\alpha\right]$$

$$= \frac{4\pi B}{\lambda r}\left[\sqrt{r^2 - z_0^2}\left(1 - \frac{z_0 dz}{r_0^2 - z_0^2}\right)\cos\alpha - z\sin\alpha - dz\sin\alpha\right]$$

Since we know the flat earth $(z = z_0)$ pattern, subtract it off to be left with the topographic term:

$$\phi_{topo} = -\frac{4\pi B}{\lambda r}\left[\frac{z_0}{\sqrt{r^2 - z_0^2}}\cos\alpha + \sin\alpha\right] dz$$

$$= \frac{-4\pi B}{\lambda r}\left[\frac{1}{\tan\Theta_0}\cos\alpha + \sin\alpha\right] dz$$

Hence the topography is linearly related to altitude, with the constant as shown above.

What does this imply? That constant-phase contours will appear at constant heights on the imaged topography. The phase will repeat every 2π radians, so therefore each "fringe" corresponds to height difference (the <u>ambiguity height</u>) defined by

$$2\pi = \frac{-4\pi B}{\lambda r}\left[\frac{1}{\tan\theta_0}\cos\alpha + \sin\alpha\right]dz$$

$$dz = \frac{2\pi}{\frac{-4\pi B}{\lambda r}\left[\frac{1}{\tan\theta_0}\cos\alpha + \sin\alpha\right]}$$

$$= \frac{-\lambda r}{2B\left[\frac{1}{\tan\theta_0}\cos\alpha + \sin\alpha\right]}$$

For our TOPSAR system, the ambiguity height is

$$dz \approx -120\text{ m}$$

so contours in phase repeat every 120 m, approximately. Note that the relation strictly holds only for a single value of r, and hence θ_0. Complete topographic reduction requires inclusion of the range dependence.

<u>Relating error to correlation</u>

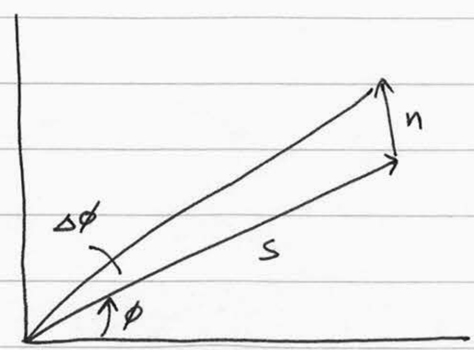
Our equation for height uncertainty gives us the height error in terms of the standard deviation of the phase σ_ϕ

$$\sigma_z = \frac{\lambda r}{4\pi B}\frac{\sin\theta}{\cos(\theta-\alpha)}\sigma_\phi$$

In order to use this expression we have to evaluate σ_ϕ either from a very good system model, or else by direct measurement from the data. The latter is often dictated because we have only the data and no accurate system model. In addition, even if we know the system well we don't a priori know the σ^0 and hence the SNR at all points in the image.

But note that we already have a measure of system performance related to phase variance – the correlation. Thus, we can express the height uncertainty in terms of the correlation we estimate as we form the interferogram. This approach allows us to easily model variable height accuracy in a scene.

Let's more carefully rederive the relationship of σ_ϕ to SNR. As before let's look at the following construction, valid for high SNR, essentially the only case of interest.



This vector plot shows that $\Delta\phi$ is

$$\Delta\phi = \frac{\text{component of } n \text{ perpendicular to } s}{|s|}$$

Since $SNR = \dfrac{\langle |s|^2 \rangle}{\langle |n_\perp|^2 \rangle + \langle |n_\parallel|^2 \rangle}$,

$$\sigma_\phi = \sqrt{\frac{1}{2 \cdot SNR}}$$ ← holds for large number of looks most accurately

and

$$\rho = \frac{1}{1 + \frac{1}{SNR}}$$

thus

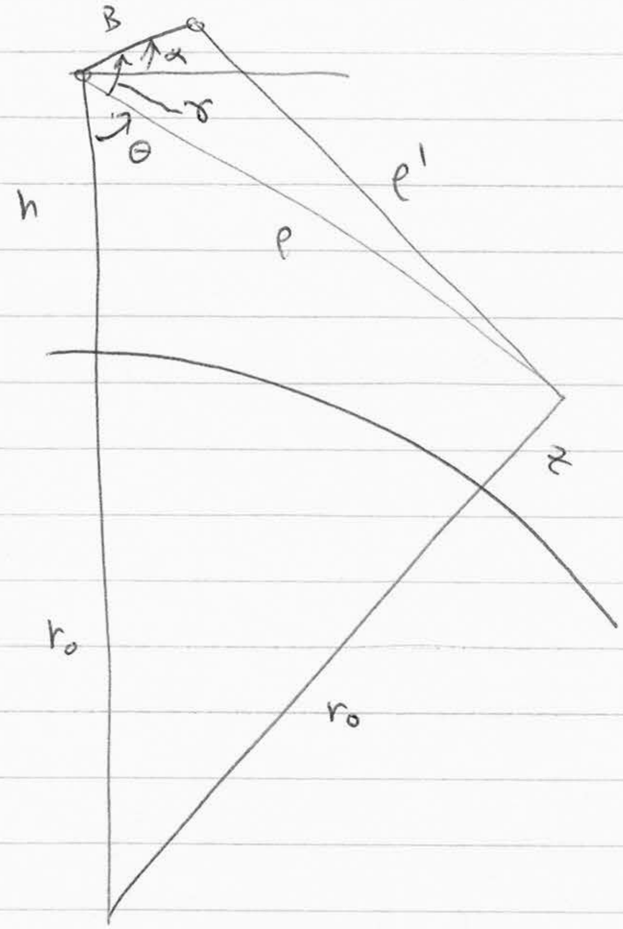
$$\sigma_\phi = \sqrt{\frac{1-\rho}{2\rho}}$$

so that

$$\sigma_z = \frac{\lambda r}{4\pi B} \frac{\sin\theta}{\cos(\theta - \alpha)} \sqrt{\frac{1-\rho}{2\rho}}$$

"Curved-Earth" phase pattern

What does the background phase from a curved planet look like in an interferogram?



The phase we measure is the difference $\rho' - \rho$. By the law of cosines

$$\rho'^2 = \rho^2 + B^2 - 2\rho B \cos\gamma$$

But $\gamma = \left(\frac{\pi}{2} - \theta\right) + \alpha$, so

$$\cos\left(\frac{\pi}{2} - \theta + \alpha\right) = \sin(\theta - \alpha)$$

so

$$\rho'^2 = \rho^2 + B^2 - 2\rho B \sin(\theta - \alpha)$$

$$\rho'^2 - \rho^2 = B^2 - 2\rho B \sin(\theta - \alpha)$$

$$\rho' - \rho \approx \frac{B^2 - 2\rho B \sin(\theta - \alpha)}{2\rho}$$

$$= \frac{B^2}{2\rho} - B \sin(\theta - \alpha) \qquad (\text{for } B \ll \rho)$$

Thus the phase as a function of θ and ρ is

$$\phi(\theta) = \frac{-4\pi}{\lambda}\left[\frac{B^2}{2\rho} - B\sin(\theta - \alpha)\right]$$

So, how does θ depend on ρ?

$$(z + r_0)^2 = (h + r_0)^2 + \rho^2 - 2\rho(h + r_0)\cos\theta$$

$$2\rho(h + r_0)\cos\theta = (h + r_0)^2 + \rho^2 - (z + r_0)^2$$

$$\cos\theta = \frac{(h + r_0)^2 + \rho^2 - (z + r_0)^2}{2\rho(h + r_0)}$$

or

$$\theta = \operatorname{acos}\left(\frac{(h + r_0)^2 + \rho^2 - (z + r_0)^2}{2\rho(h + r_0)}\right)$$

and the curved earth pattern for no topography is this equation with $z = 0$.