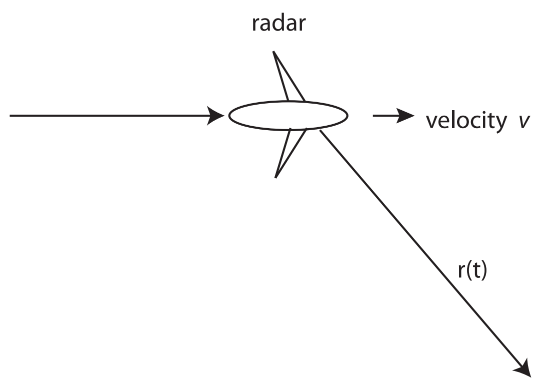
The phase history of a point

We have seen how we can exploit the frequency change of a radar echo with time to improve its resolution over the real aperture radar case. This resulted ion what we called an "unfocused" processor. By this terminology you might guess that there is an alternative called a "focused" processor, which we will discuss now.

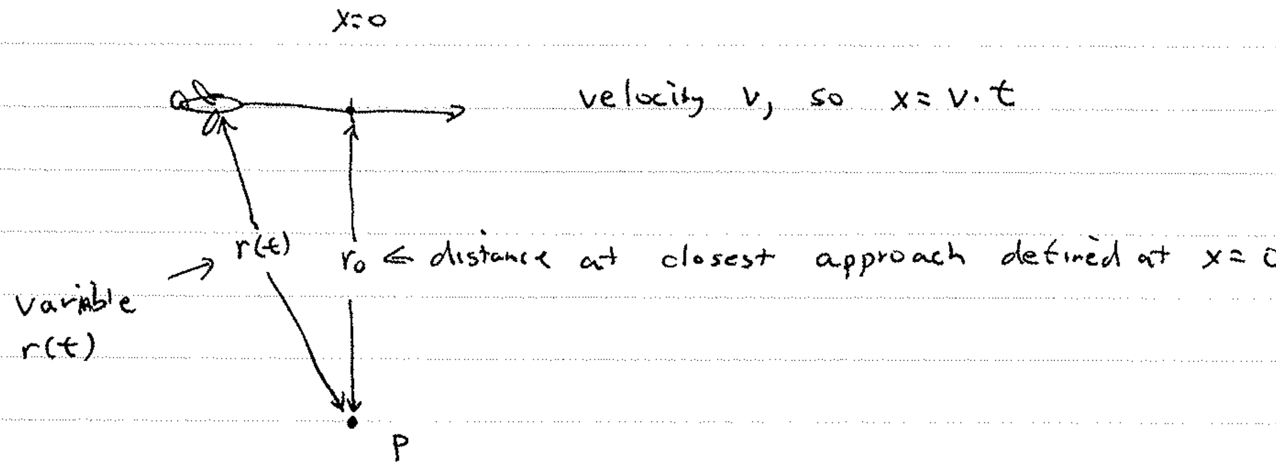
We begin by studying the variation of the phase of a radar echo as our platform flies by. Here's the geometry:



so the phase is simply

$$\phi(t) = \frac{-4\pi}{\lambda} r(t).$$

We can derive an expression for r(t) as follows:



Note $r_0 = r(0)$.

Since by Pythagoras

$$r^2(t) = r_0^2 + x^2$$

$$r^2(t) = r_0^2 + v^2 t^2$$

$$r(t) = \sqrt{r_0^2 + v^2 t^2}$$

Now, we only need consider values of x (or vt) for points in the illuminated part of the surface, that is within the antenna beam. We know that limit:

$$-\frac{r_0 \lambda}{2\ell} < x < \frac{r_0 \lambda}{2\ell}$$

Since in most cases the antenna length ℓ is much greater than the radar wavelength, we can assume

$$x^2 << r_0^2$$

We can simplify our range vs. time equation thusly:

$$r(t) = r_0 \sqrt{1 + \frac{v^2 t^2}{r_0^2}}$$

or

$$r(t) \approx r_0 \left(1 + \frac{1}{2}\frac{v^2 t^2}{r_0^2}\right)$$

$$r(t) \approx r_0 + \frac{1}{2}\frac{v^2 t^2}{r_0}$$

Hence

$$\phi(t) = -\frac{4\pi}{\lambda}\left[r_0 + \frac{1}{2}\frac{v^2 t^2}{r_0}\right]$$

Neglecting the constant phase term,

$$\phi(t) \approx -\frac{2\pi}{\lambda}\frac{v^2}{r_0} t^2$$

Note that the phase is quadratic in time. Where have we seen that before?

> Thus we see that the azimuth response of the radar echo is also a chirp function.

What are the equivalent frequency characteristics of this phase history?

$$f(t) = \frac{1}{2\pi}\phi'(t)$$

$$= -\frac{2v^2}{\lambda r_0} t$$

and for the chirp slope s

$$s = \frac{1}{2\pi}\phi''(t)$$

$$= -\frac{2v^2}{\lambda r_0}$$

In the azimuth direction, we often use the term <u>chirp rate</u> rather than chirp slope to distinguish it from the range modulation.

Knowing the chirp slope/rate, we can thus apply the same kind of matched filter in azimuth that we used in the range processing operation. We can define a reference signal and correlate it against the distribution of scatterers in azimuth to achieve resolution in azimuth.

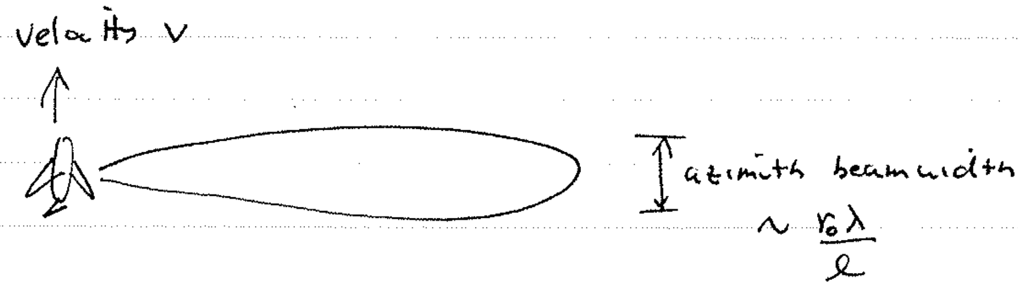
<u>Azimuth resolution relation</u>

Now we can evaluate the resolution achievable by matched filter processing in azimuth. We know that the resolution will be the reciprocal of bandwidth, as it was for range processing.

In range, we obtained bandwidth by the following:

$$Bw = s \cdot \tau$$

We know the chirp rate, but what is our equivalent value of the pulse length? It is simply the time that the target is illuminated by an antenna.



So for velocity v and beamwidth $\frac{r_0 \lambda}{\ell}$,

$$\tau_{az} = \frac{r_0 \lambda}{v \ell}$$

Hence our bandwidth in Hertz is

$$BW_{az} = \frac{-2v^2}{\lambda r_0} \cdot \frac{r_0 \lambda}{v \ell}$$

$$= \frac{-2v}{\ell}$$

which means our resolution in seconds is (ignoring the - sign)

$$\delta_t = \frac{\ell}{2v}$$

and the resolution in meters is just $v \cdot \delta_t$:

$$\boxed{\delta_{az} = \frac{\ell}{2}}$$ ← A very surprising result!

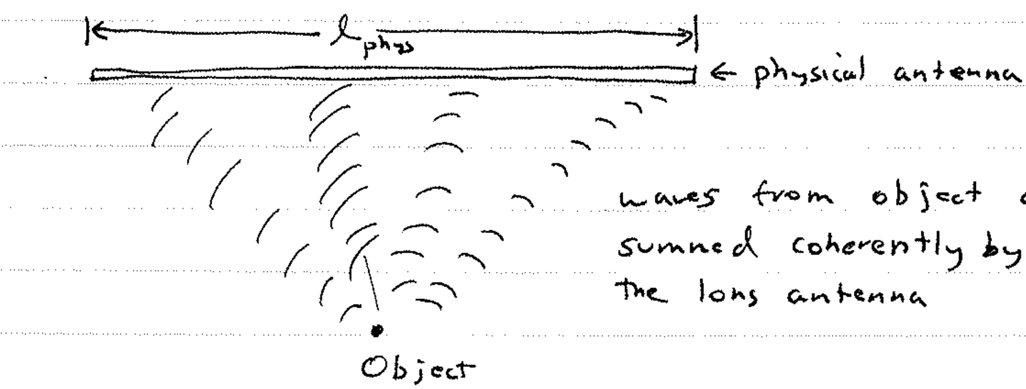
The azimuth resolution in meters is independent of:

Wavelength
Velocity
Range
→ Everything but antenna length!

Clearly this azimuth matched filtering is a powerful technique. Forming images using this approach is called "synthetic aperture radar", because we can understand also as a synthetic beamforming problem.

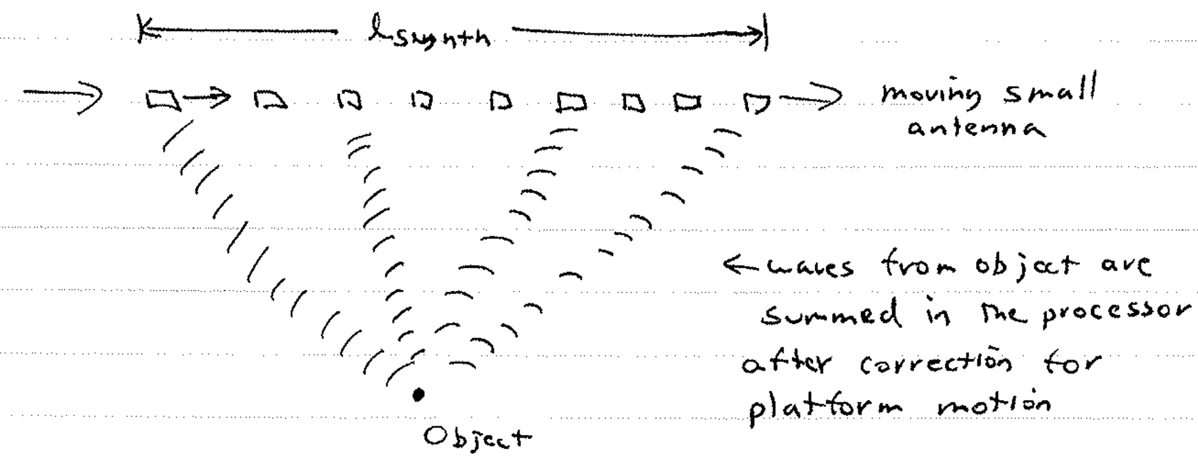
Synthetic beam viewpoint

Long physical antenna



$$\text{angular resolution} = \frac{\lambda}{l_{phys}}$$

Moving platform, pulsing:



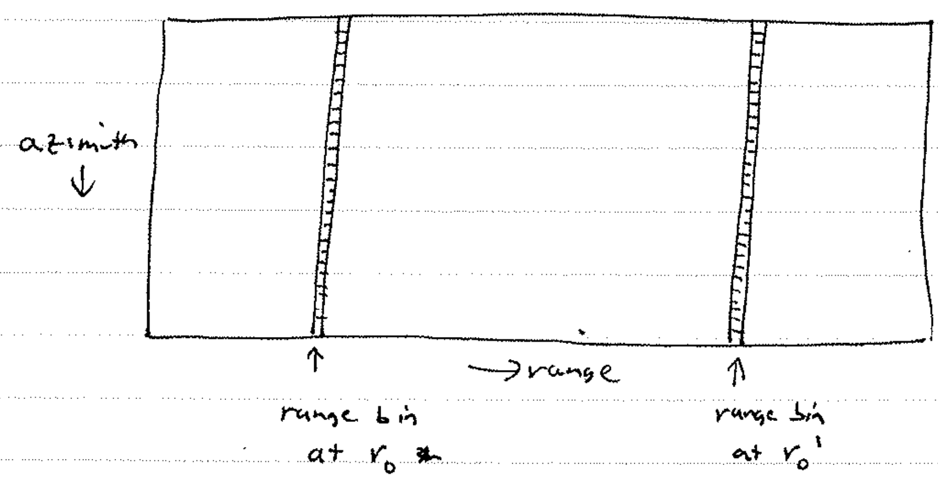
$$\text{angular resolution} = \frac{\lambda}{l_{synth}}$$

The length of the synthetic aperture l_{synth} depends on how wide the beamwidth of the small, moving antenna is.

Also, in the radar case we obtain an extra factor of two in resolution due to two-way travel.

SAR processing algorithm

Consider a range-compressed image:



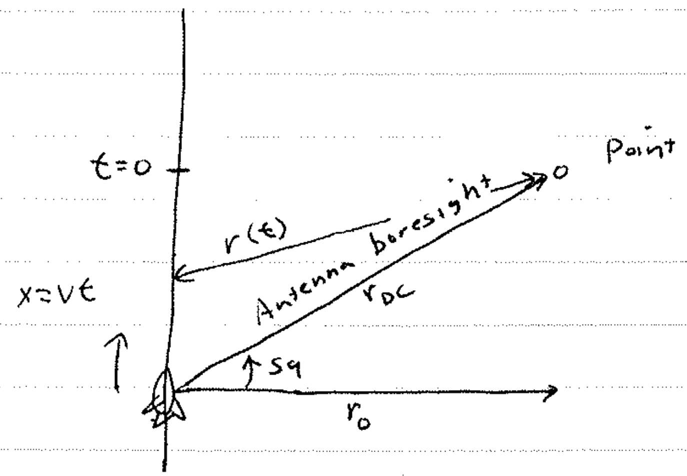
Two highlighted columns are shown. Each represents a sequence of complex azimuth samples which we can match filter to derive azimuth resolution of $\frac{\ell}{2}$. Our azimuth processing algorithm simply reads in samples down from each column in the matrix, applies the appropriate reference function, and outputs the correlation result.

Note, though, that because the two bins are at different ranges, the reference matched filters are different, in that the chirp slopes are inversely proportional to range.

Hence, a new reference function is needed for each azimuth bin, rather than using the same one over again as we did in the range processing case.

Squinted geometries

As in the unfocused case, the radar may not be pointed at boresight, hence the majority of the energy may not come from the zero Doppler location. Let's look at a squinted case and see how that changes the phase history.



Let's let the subscript DC denote Doppler centroid, the time or location when the target point goes through the antenna boresight. Hence, r_{DC} is the range from the object to the radar at this instant, and so forth.

In this situation, rather than being symmetrical about zero Doppler, the azimuth frequency spectrum is shifted by a frequency equal to the Doppler centroid, f_{DC}. To first order the bandwidth remains the same.

What are the parameters of the phase history in this geometry?

We still have the exact relation

$$r^2(t) = r_0^2 + V^2 t^2$$

Note also that in this coordinate system the composite squint angle sq is related to the Doppler centroid by

$$f_{DC} = -\frac{2v}{\lambda} \frac{x_{DC}}{r_{DC}} = \frac{-2v^2 t_{DC}}{\lambda r_{DC}}$$

because for X_{DC} or $t_{DC} < 0$ we have a positive Doppler:

$$\sin sq = \frac{-X_{DC}}{r_{DC}} \quad \text{(positive squint defined as forward looking)}$$

Noting that $r_{DC}^2 = r_0^2 + X_{DC}^2$

$$= r_0^2 + V^2 t_{DC}^2 ,$$

we have

$$r^2(t) = r_{DC}^2 - v^2 t_{DC}^2 + V^2 t^2$$

Now, define a new time variable $t' = t - t_{DC}$, centered at the time the object is aligned with the antenna boresight. Now

$$r^2(t) = r_{DC}^2 - V^2 t_{DC}^2 + V^2 (t'^2 + 2 t_{DC} t' + t_{DC}^2)$$

$$= r_{DC}^2 + V^2 t'^2 + 2 V^2 t_{DC} t'$$

and using the same expansion we used before

$$r(t) \approx r_{DC} \left(1 + \frac{1}{2} \frac{V^2 t'}{r_{DC}^2} + \frac{V^2 t_{DC} t'}{r_{DC}^2}\right)$$

Then, neglecting the constant ~~phase~~ term again,

$$\phi(t) = -\frac{4\pi}{\lambda}\left(\frac{1}{2}\frac{v^2 t'^2}{r_{DC}} + \frac{v^2 t_{DC} t'}{r_{DC}}\right)$$

$$= -\frac{2\pi}{\lambda}\frac{v^2}{r_{DC}} t'^2 - \frac{4\pi v^2 t_{DC}}{\lambda r_{DC}} t'$$

quadratic term as before but at range r_{DC} rather than r_0

linear term corresponding to frequency offset at Doppler centroid

As before, our chirp frequency is

$$f = \frac{\phi'(t)}{2\pi}$$

$$= -\frac{1}{2\pi}\cdot\frac{2\pi}{\lambda}\cdot\frac{v^2}{r_{DC}}\cdot 2t' - \frac{2v^2}{\lambda r_{DC}} t_{DC}$$

At $t' = 0$ (the point of ~~close~~ boresight alignment)

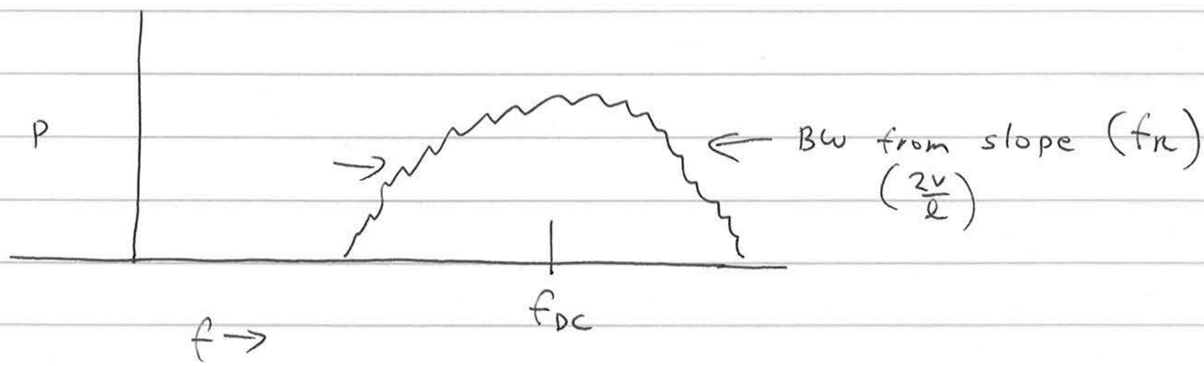
$$f = \frac{-2v^2 t_{DC}}{\lambda r_{DC}} = f_{DC}$$

$$= \frac{2v}{\lambda}\sin(sq)$$

The chirp rate (f_R) is

$$f_R = \frac{\phi''(t)}{2\pi} = \frac{-2v^2}{\lambda r_{DC}}$$

Hence a plot of the received chirp spectrum is



Note the time in the beam follows from the velocity and antenna length

$$\tau_{\text{in beam}} = \frac{\frac{r_{DC}\lambda}{l}}{v} = \frac{r_{DC}\lambda}{l v}$$

$$BW_{at} = \tau_{\text{in beam}} \cdot f_R$$

$$= \frac{r_{DC}\lambda}{l v} \cdot \frac{2v^2}{r_D \cdot \lambda} = \frac{2v}{l}$$

$\Rightarrow$ minimum prf $= \dfrac{2v}{l}$ $\leftarrow$ doesn't depend on λ or r

Unfocussed SAR design

1. Size of projected beamwidth on ground:

$$\frac{r_o \lambda}{l} = \frac{15000 \times 0.06}{1} = 900 \text{ m}$$

2. Cycle time between bursts: $(v = 200 \text{ m/s})$

$$\text{cycle time} = \frac{900 \text{ m}}{200 \text{ m/s}} = 4.5 \text{ s}$$

3. Range of Doppler frequencies

$$f_m = \frac{2v}{\lambda} \sin\theta \sin\phi_{max} = \frac{2v}{\lambda} \sin(\varphi_{max})$$

$$= \frac{2v}{\lambda} \frac{x_{max}}{r}$$

$$f_{max} = \frac{2 \cdot 200}{0.06} \times \frac{450 \text{ m}}{\cancel{15000 \text{ m}}\ 15000 \text{ m}} = 200 \text{ Hz}$$

$\Rightarrow$ bandwidth $= 400$ Hz

4. How many pulses?

$$\delta_{az} = \sqrt{\lambda R}$$

$$= \sqrt{0.06 \times 15000}$$

$$= 30 \text{ m}$$

1 resel $= 30$ m Projected beam $= 900$ m

$\Rightarrow$ # bins in frequency $= \frac{900}{30} = \underline{30}$

slightly oversample, use 32 bins (pulses)

5. Check design:

32 pulses @ 400 Hz $\Rightarrow \frac{32}{400} = 0.08$ seconds

$0.08\,s \times 200\,m/s = 16\,m$ (less than 30 m, so ok)

32 pulses is ok system.

Suppose we used 64 pulses (oversample by 2)

$\frac{64}{400} = 0.16\,s \qquad 0.16 \times 200 = 32\,m$ (slightly more than 30 m)

so this will slightly blur output image.

Alternate design method:

1. $\delta_{az} = \sqrt{rx} = 30$ m

2. pulse spacing is $\frac{200 \text{ m/s}}{400 \text{ Hz}} = 0.5$ m

$\Rightarrow$ burst size $= \frac{30}{0.5} = 60$ or 64 as a power of two

3. ~~$x_{max} = 45$~~

~~projected antenna size =~~

frequency bin size $= \frac{400}{64} = 6.2$ Hz

ground spacing for output:

$$6.2 \text{ Hz} = \frac{2v}{\lambda} \frac{x}{r} = \frac{2 \cdot 200}{0.06} \cdot \frac{x}{15000}$$

$\Rightarrow$ 13.9 m / pulse in output map