The Radar Equation

The quality of a radar measurement depends on the combination of radar system parameters, imaging geometry, and surface characteristics for each observation. These can be related in a quantitative sense using the radar equation. There is not a single radar equation in universal use, rather it refers to a family of equations, only one of which may be appropriate for a given situation. Thus we may see in the literature equations such as

$$\begin{aligned}\mathrm{SNR_o} &= P_s/kF_{op}T_sB_n \\ &= P_tG^t\sigma A_e/[(4\pi R^2)^2kF_{op}T_sB_n]\end{aligned}$$

or

$$\mathrm{SNR_o} = \frac{P_t\int[G^t(\theta,\phi)\sigma^0(\theta,\phi)A_e(\theta,\phi)/(4\pi R^2)^2]\,\mathrm{d}A}{F_{op}kT_sB_n}$$

or
$$\mathrm{SNR_o} = P_t\lambda^2G^2\sigma^0\delta x\delta R_g/[(4\pi)^3R^4F_{op}kT_sB_n]$$

or
$$\mathrm{SNR_o} = [P_t\sigma^0\lambda^2/(4\pi)^3F_{op}kT_sB_n]\int[G^2(\theta,\phi)/R^4]\,\mathrm{d}A$$

or
$$\mathrm{SNR_o} = P_tG^2\sigma^0\lambda^2\theta_Hc\tau_p/[2(4\pi R)^3F_{op}kT_sB_n\sin\eta]$$

or
$$\mathrm{SNR}^1 = P_{av}G^2\lambda^3\sigma^0\delta R_g/[2(4\pi R)^3V_{st}F_{op}kT_s]$$

and these are all from the same chapter in our textbook!

Clearly it make no sense to memorize this gobbledegook. We will instead look at the derivation for the general case, which we can specialize in time as needed.

Signal-to-noise ratio

One common feature of all versions of the radar equation is that they yield an estimate of quality of a radar measurement in terms of the signal-to-noise ratio. Consider this graph of the received power in a radar system vs. time:

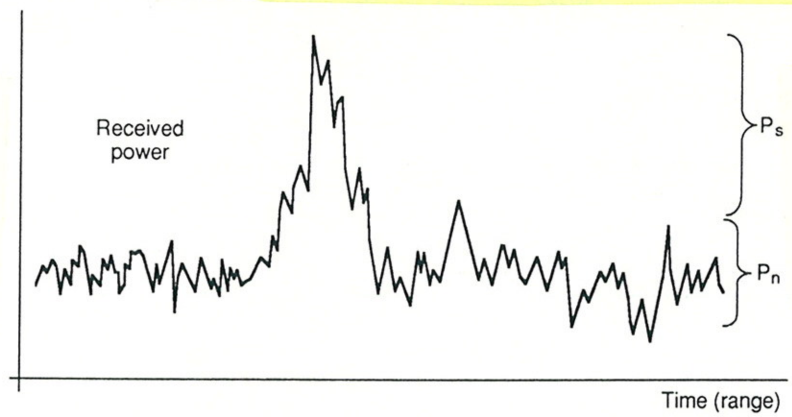


Figure 2.2 The receiver signal to noise ratio $SNR_o = P_s/P_n$.

← signal plus noise level

← noise level

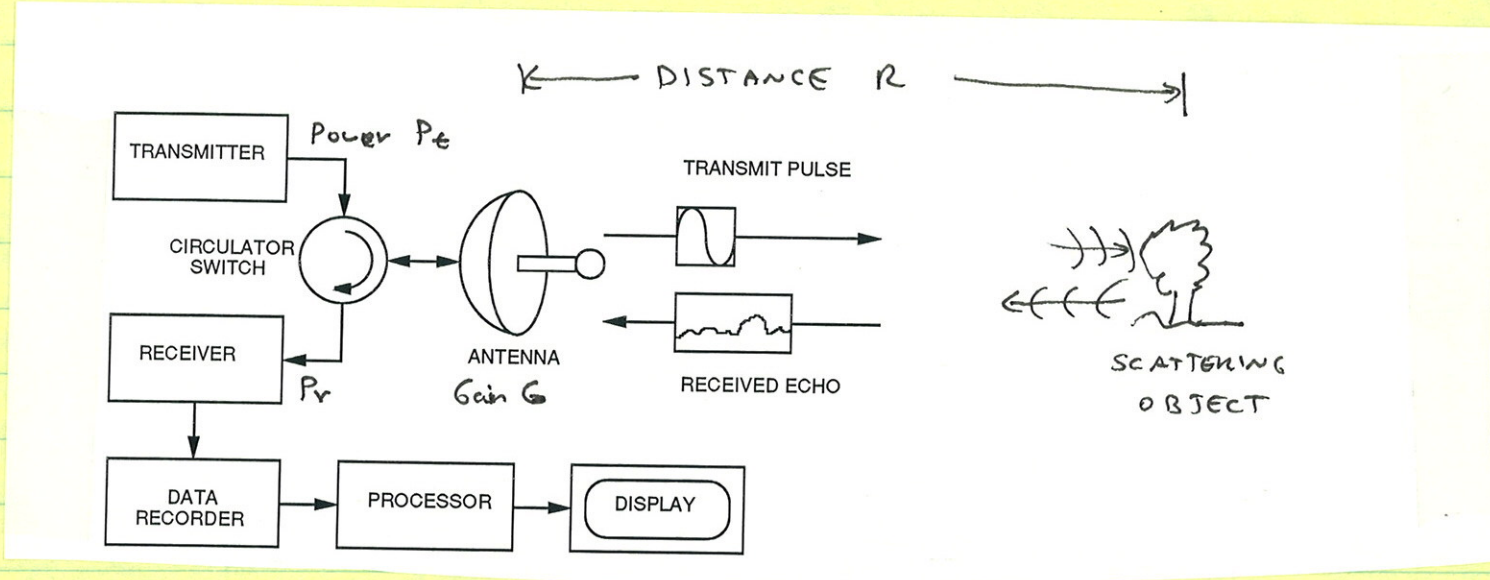
The power received in the absence of a scattering object is the "noise level", denoted P_n. If there is a scatterer in the beam, to this we add some signal power P_s so the total power received is $P_s + P_n$. We can then define the signal to noise ratio (SNR) as

$$SNR = \frac{P_s}{P_n}$$

For an imaging radar, we try to achieve SNR's of at least 10, while 100 - 1000 is sometimes obtainable. How can we calculate this quantity from system and geometric parameters?

Derivation of Radar Equation:

We begin by considering the radar block diagram shown last time:



Suppose the transmitter generates a pulse with transmit power P_t. Since there are 4π steradians on the surface of a sphere, if this power were isotropically radiated the power density on the surface of a sphere of radius equal to the distance from the radar to the scatterer R would be

$$\text{Power density} = \frac{P_t}{4\pi R^2} \quad w/m^2$$

In other words, the power would be uniformly spread out over the sphere.

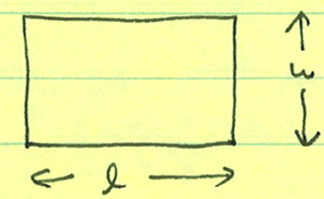
In this situation power is not used very efficiently, since the object we want to measure fills only a very small fraction of the total surface area. Hence we use an antenna which concentrates most of the energy in a single direction, ideally toward the object. The degree of concentration is the directivity gain G of the antenna.

We won't be discussing details of antenna theory here, but need to use certain relations that allow us to calculate gain from antenna parameters. In particular, we use for now

$$G = \frac{4\pi A}{\lambda^2}$$

where A is the antenna area and λ is the transmitted wavelength.

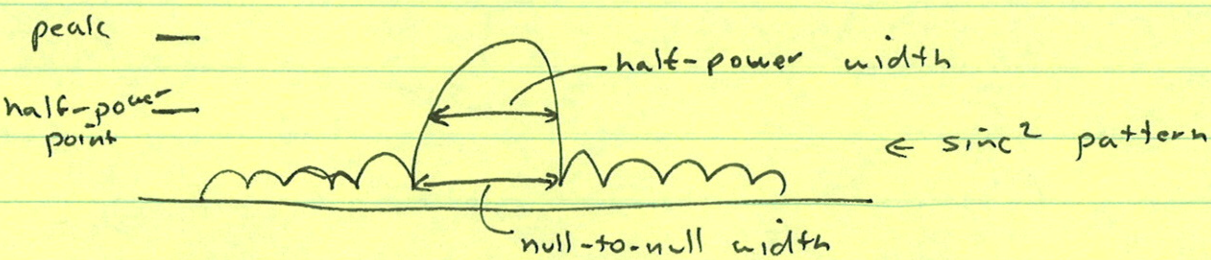
Where does this come from? Consider a rectangular antenna aperture of length l and width w:



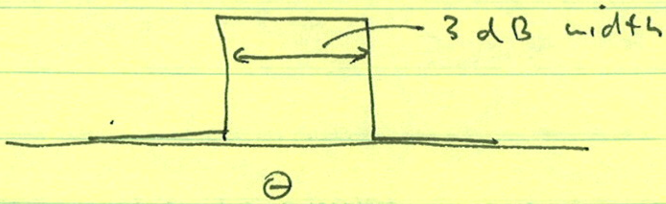
From Fourier antenna theory, the antenna pattern from this aperture is

$$\operatorname{sinc}^2\left(\frac{l}{\lambda}\theta_l\right)\operatorname{sinc}^2\left(\frac{w}{\lambda}\theta_w\right)$$

where θ_l and θ_w are angles in the l- and w- directions and we have ignored normalization factors for now. In other words the power pattern is a sinc^2 function in each dimension.



The null-to-null width of the sinc^2 pattern is $\frac{2\lambda}{l}$ or $\frac{2\lambda}{w}$, depending on which dimension we are using. It is customary to define an "equivalent width" of an antenna as the half-power or "3 dB" width, and model the system as if the pattern were constant over that portion and zero elsewhere:



For an unweighted sinc^2 pattern, the 3dB width Θ_{3dB} is

$$\Theta_{3db} \approx 0.89 \cdot \frac{\lambda}{l}$$

or simply $\Theta_{3dB} \approx \frac{\lambda}{l}$

Now, back to gain equation. The angular area defined by the antenna is

$$\text{angular area} = \frac{\lambda}{l} \cdot \frac{\lambda}{w}$$

$$= \frac{\lambda^2}{A} \quad \text{steradians}$$

where A is the antenna area. The gain is the ratio of the the total angular area of a sphere to that where the energy is concentrated. Since 4π steradians make up a sphere,

$$G = \frac{4\pi}{\lambda^2/A} = \frac{4\pi A}{\lambda^2}$$

So if we want to calculate the power density at our scattering object, we multiply the isotropic value by the antenna gain:

$$\text{Power density} = \frac{P_t}{4\pi R^2} \cdot G$$

$$\text{Power density} = \frac{P_t}{4\pi R^2} \cdot \frac{4\pi A}{\lambda^2} \quad (w/m^2) \quad \text{or} \quad \frac{P_t}{4\pi R} \cdot G \quad (w/m^2)$$

Now, how much power is actually intercepted by our object? If the scatterer has an area A_{scat}, the power intercepted is

$$\text{Intercepted power} = \frac{P_t}{4\pi R^2} \cdot G \cdot A_{scat} \quad (w)$$

Only a fraction of this power is actually returned in the direction of the radar. This fraction, called the <u>normalized radar cross-section</u>, is usually denoted $\underline{\sigma^0}$ and called sigma-zero. This power is now modeled as isotropically radiated from the scatterer, so the reflected power density back at the receiving antenna

$$\text{Power density at receiving antenna} = \frac{P_t}{4\pi R^2} \cdot G \cdot A_{scat} \cdot \sigma^0 \cdot \frac{1}{4\pi R^2}$$

Finally, if the antenna area is A the total power at the receiver is

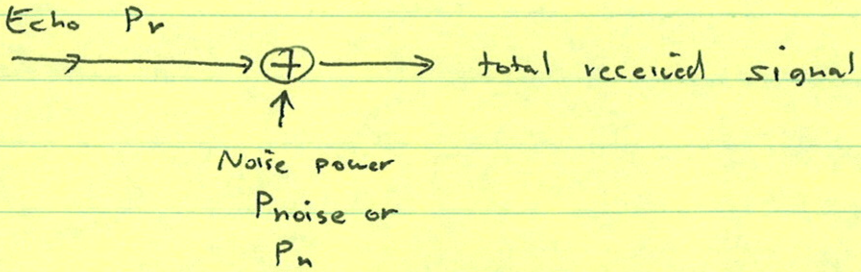
$$P_r = \frac{P_t}{4\pi R^2} \cdot G_t \cdot A_{scat} \cdot \sigma^0 \cdot \frac{A_r}{4\pi R^2}$$

where we have introduced subscripts G_t and A_r for generality in case separate antennas are used for transmit and receive.

This is the general <u>radar equation</u>.

That takes care of the signal power calculation. We are after signal-to-noise ratio, however.

Our noise model here will be very simple: we model the receiver as having an equivalent noise temperature T_{sys} -



where $P_n = k T_{sys} B$; T_{sys} is the noise temperature, B is the system banwidth in Hertz, and k is Boltzmann's constant 1.38×10^{-23} J/K. In fact this simple model only holds if all impedances are matched, which we'll assume here. Hence our SNR equation becomes

$$SNR = \frac{\frac{P_t}{4\pi R^2} \cdot G_t \cdot A_{scatt} \cdot \sigma^0 \cdot \frac{A_r}{4\pi R^2}}{k T_{sys} B}$$

I prefer to memorize the above form because I can readily rederive it, but you can group terms to get something like

$$SNR = \frac{P_t G_t A_r A_{scatt} \sigma^0}{(4\pi R^2)^2 k T_{sys} B}$$

which is what is done in much of the literature. Substituting relations like $G = \frac{4\pi A}{\lambda^2}$ provide the other forms you see.

Designing a radar using the radar equation

Let's now look at how we might actually design a radar using the radar equation. Let's say we want to be able to detect an oak tree at a distance of 10 km with an SNR of 100. The tree is 20 m tall and 8m wide. We'll assume that the σ^0 of the tree is 0.1, and that the radar operates at L-band with a wavelength of 25 cm. Also assume that the bandwidth of the receiver is 1 MHz.

We could simply plug these numbers into the radar equation, and after the appropriate multiplications and divisions infer some of the various other system parameters. However, to increase the intuitiveness of the process, and also to develop a very useful design tool, let's represent the parameters in log space using decibels so we can add and subtract numbers instead. In this way a simple spreadsheet can be used for radar design.

For review, a ratio is represented in decibels as

$$dB = 10 \cdot \log_{10} R$$

where R is the value in linear space and dB is its representation in dB.

Some simple examples:

R	dB
1	0
100	20
31.6	15

Now, organize the quantities in a <u>design control table</u> as follows:

Signal parameters			Noise parameters		
Param	Value	dB	Param	Value	dB
Transmit power	1000 w	30			
Antenna gain	50	17	Boltzmann	1.38×10^{-23}	-229.
$\frac{1}{4\pi}$		-11	Tsys	1000 K	30
$1/R^2$	10 km^2	-80	Bandwidth	1 MHz	60
A_{scatt}	160 m^2	22			
σ^o	0.1	-10			
A_r	0.248	-6			
$\frac{1}{4\pi}$		-11			
$1/R^2$	10 km^2	-80			
Signal power		-129 dB	Noise power		-139

Notes: $A_r = \frac{6\lambda^2}{4\pi}$

$SNR = P_r - P_n = -129 - (-139) = +10\,dB$ or $10:1$.

We wanted an SNR of 100 or 20 dB. We have only 10 dB in our present design. We'll have to get another 10 dB out of the system.

The values we can change are the transmit power, antenna gain, and noise temperature. All others were given to us. We can either increase one by the full 10 dB or increase each a little.

It's probably easier to change each slightly. Let's up the transmit power by a factor of ~3 (5 dB), and decrease the noise temperature by about 3, another 5 dB. Our system will then have the desired response.

The system we have designed, then, has the following parameters:

Transmit power:	3 kW
Antenna area:	0.248 m^2
Wavelength:	25 cm
Bandwidth:	1 MHz
Noise temperature:	333 K

These parameters permit measurement of our tree with an SNR of 20 dB at a distance of 10 km.

A more complete design control table

A realistic design control table would have additional entries to those we listed above. For example, antennas are not 100% efficient, so an entry for this for both transmit and receive is needed. Also quite often there are reasons to specify individually the antenna length and width, as we'll see next time. A more complete table for our system might eventually look more like:

Signal to Noise Ratio dB Table

	Value	dB
Wavelength	0.250	
Peak transmit power	3000	34.8
Cable loss	0.7	-1.5
Antenna efficiency	0.5	-3.0
Antenna gain		17.0
1/4pi		-11.0
1/r^2	10000	-80.0
Scattering area	160	22.0
Sigma zero	0.1	-10.0
1/4pi		-11.0
1/r^2	10000	-80.0
Antenna width	0.25	-6.0
Antenna length	1	0.0
Antenna efficiency	0.5	-3.0
Cable losses	0.7	-1.5
Signal power		-133.3
Noise temperature	333	25.2
Boltzmann's constant	1.38E-23	-228.6
Noise bandwidth	1.00E+06	60.0
Noise power		-143.4
Signal to noise ratio		**10.1**

This was implemented using an Excel spreadsheet.